A new 1D parameter-control chaotic framework

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ABSTRACT

This paper introduces a novel parameter-control framework to produce many new one-dimensional (1D) chaotic maps. It has a simple structure and consists of two 1D chaotic maps, in which one is used as a seed map while the other acts as a control map that controls the parameter of the seed map. Examples and analysis results show that these newly generated chaotic maps have more complex structures and better chaos performance than their corresponding seed and control maps.

Keywords: parameter-control, framework, chaotic map

1. INTRODUCTION

The chaotic map is a type of dynamic systems. It has the property of unpredictability and can generate different pseudo-random sequences with infinitesimally different settings of initial values and control parameters. With these significative properties, the chaotic map is a good candidate in many applications. In recent years, chaotic maps have been successfully used in economics,[1] population dynamics,[2] security applications[3,4] and so on. Especially when used in security applications, compared with other security applications,[5–8] the chaotic map-based applications show high performance.

When more and more chaotic maps are used in different applications, there come two obvious weaknesses of existing 1D chaotic maps. The first weakness is that the trajectories of existing 1D chaotic maps are easy to be predicted because they usually have simple structures.[9] The other is that the quality of pseudo-random sequences generated by existing 1D chaotic maps is too low because their chaos performance is usually not so good.[10] Therefore, developing new chaotic maps with more complex structures and better chaos performance is necessary. In the application of image encryption, some new chaotic maps have been developed.[11,12] These applications have proved that the new chaotic maps can improve the image security level when used in image encryption.

In this paper, we propose a new parameter-control chaotic framework with a simple structure. Using this framework, new 1D chaotic maps can be generated from any combination of two 1D chaotic maps. Examples and performance comparisons are provided to demonstrate the performance of our proposed framework.

The rest of this paper is organized as follows: Section 2 will briefly review several traditional 1D chaotic maps. In Section 3, the parameter-control chaotic framework will be proposed and discussed. Section 4 will introduce three examples of new chaotic maps and their chaos performance will be discussed in Section 5. Section 6 will give a conclusion.

2. BACKGROUND

In this section, three 1D chaotic maps will be reviewed, which will be used to generate new chaotic maps in Section 4.

2.1 Tent map

The Tent map is a dynamic system that does the operations of folding and stretching. When the input value is smaller than 0.5, it will stretch the value. Otherwise, it will firstly fold the value into range of [0, 0.5], and then stretch it into range of [0, 1]. By iterating the procedure, any initial point in range of [0, 1] will generate a sequence x_n in [0, 1]. Mathematically, the Tent map is defined as Eqn. (1)

$$x_{n+1} = \begin{cases} ux_n & \text{for } x_n < 0.5 \\ u(1-x_n) & \text{for } x_n \geq 0.5 \end{cases} \tag{1}$$

where u is a parameter and $u \in$[0, 2]. When the parameter $u \in$(1, 2], the Tent map will be chaotic. The bifurcation diagram and few initial iteration functions of the Tent map are shown in Fig. 1(a). From its bifurcation diagram, we can see that, when u is close to 2, the Tent map's chaos performance becomes better.

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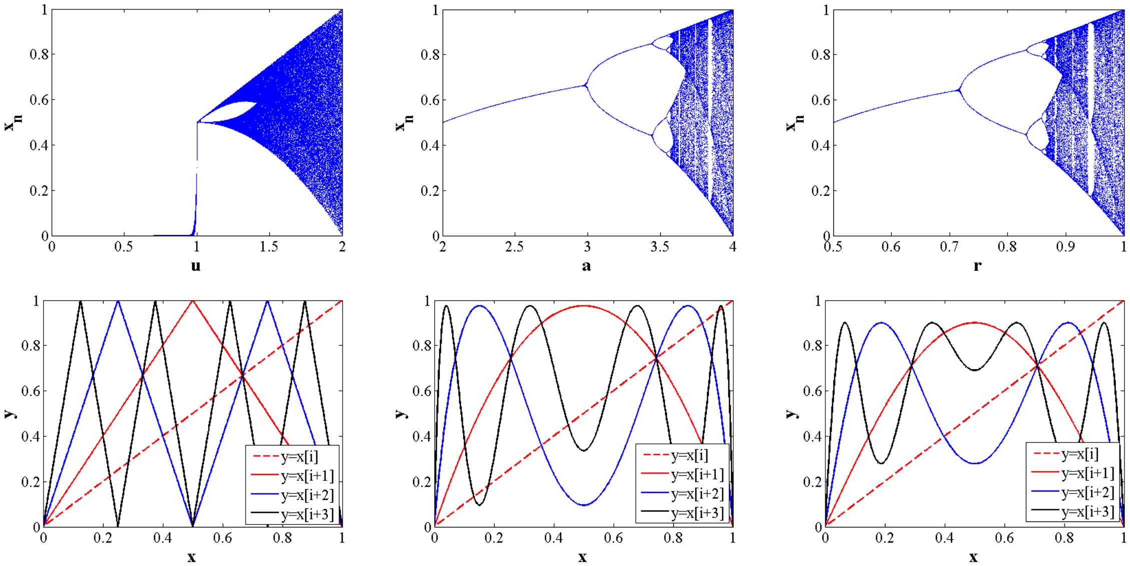

(a) (b) (c)

Figure 1: The chaos performance of three traditional 1D chaotic maps. The first and second rows are the bifurcation diagrams and iteration functions of (a) the Tent map; (b) the Logistic map and (c) the Sine map, respectively.

2.2 Logistic map

The Logistic map is a polynomial map widely used in many applications. Mathematically, the Logistic map is defined as Eqn. (2)

$$x_{n+1} = ax_n(1 - x_n) \tag{2}$$

where a is a parameter and $a \in$[0, 4]. x_n are the iteration values in range of [0, 1]. The bifurcation diagram and few initial iteration functions of the Logistic map are shown in Fig. 1(b). As can be seen from its bifurcation diagram, when $a \in$[3.57, 4] (approximately), the Logistic map is chaotic.

2.3 Sine map

The Sine map comes from the sine curve. Its chaotic behavior is similar with that of the Logistic map, but its mathematic representation is totally different, which is defined as Eqn. (3)

$$x_{n+1} = r\sin(\pi x_n) \tag{3}$$

where r is a parameter for the Sine map and $r \in$[0, 1]. The iteration input/output values x_n are in the range of [0, 1]. When the parameter $r \in$[0.867, 1], the Sine map has chaotic behaviors. The bifurcation diagram and few initial iteration functions of the Sine map are shown in Fig. 1(c). From the bifurcation diagrams of the Logistic and Sine maps, we can see that their chaotic behaviors are similar.

3. NEW PARAMETER-CONTROL CHAOTIC FRAMEWORK

In this section, a new parameter-control chaotic framework is proposed. Using this framework, new 1D chaotic maps can be generated from any two 1D chaotic maps. The proposed framework uses the output of one chaotic map to control the parameter of another chaotic map. Its structure is shown in Fig. 2. As can be seen, the 1D chaotic map $G(x)$ is used as a seed map while another 1D chaotic map $F(x)$ acts as a control map, which controls the parameter of the seed map. The 'Linear scaling' operation is to ensure the parameters within $G(x)$'s chaotic range.

The mathematic representation of the framework is defined by Eqn. (4)

$$x_{n+1} = G(p'_{n+1}, x_n) \tag{4}$$

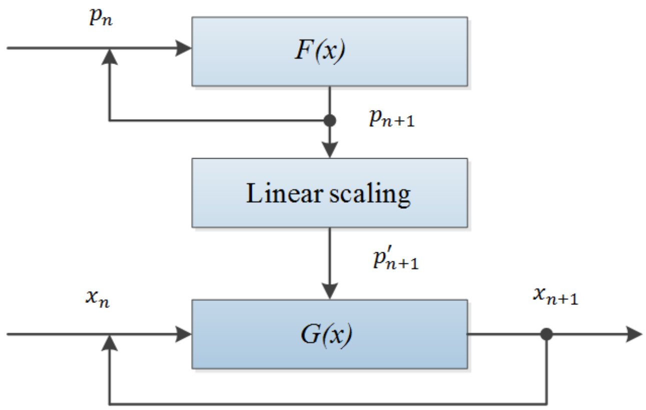

Figure 2: The proposed parameter-control chaotic framework.

where $G(x)$ is the seed map which is an existing 1D chaotic map. x_n are the iteration values of the new map and p'_{n+1} are parameters defined by Eqn. (5)

$$p'_{n+1} = L(p_{n+1}) \tag{5}$$

where $L(x)$ is a linear function that ensures the p'_{n+1} within $G(x)$'s chaotic range. p_{n+1} are generated by the control map which are defined as Eqn. (6)

$$p_{n+1} = F(p_n) \tag{6}$$

in which the control map $F(x)$ is also an existing 1D chaotic map.

The traditional 1D chaotic maps have a fixed parameter setting for all iterations. Using a 1D chaotic map to control the parameter of the seed map, the proposed framework utilizes a dynamic parameter in each iteration to generate new chaotic maps. This means that the newly generated chaotic maps have more complex structures and unpredictable output sequences than traditional 1D chaotic maps. Therefore, the new chaotic maps generated by the proposed framework are suitable for different applications such as information security.

4. EXAMPLES OF NEW CHAOTIC MAPS

Using different 1D chaotic maps as the seed and control maps in the proposed framework, new 1D chaotic maps can be generated. In this section, three examples of new chaotic maps will be introduced to show the excellent properties of the proposed framework.

4.1 The Tent-control Tent (TCT) map

In the proposed framework, the seed and control maps can be a same 1D chaotic map. When the seed and control maps are both Tent maps, a new 1D chaotic map can be generated, called the Tent-control Tent (TCT) map. The structure of the TCT map is shown in Fig. 3. A Tent map is used to control the parameter of another Tent map to generate iteration values. By this way, the seed map's parameter is no longer a fixed one and changes in each iteration.

The mathematic representation of the TCT map is defined by Eqn. (7)

$$x_{n+1} = \begin{cases} u'_{n+1}x_n & \text{for } x_n < 0.5 \\ u'_{n+1}(1-x_n) & \text{for } x_n \geq 0.5 \end{cases} \tag{7}$$

where x_n are iteration values and u'_{n+1} are parameters which are defined as Eqn. (8)

$$u'_{n+1} = 1.8 + 0.2u_{n+1} \tag{8}$$

in which u_{n+1} are iteration values of control map defined in Eqn. (9)

$$u_{n+1} = \begin{cases} uu_n & \text{for } u_n < 0.5 \\ u(1-u_n) & \text{for } u_n \geq 0.5 \end{cases} \tag{9}$$

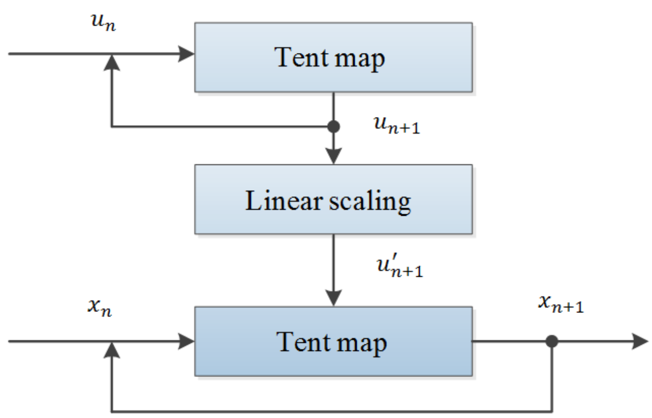

Figure 3: The structure of the TCT map.

where u is a parameter of control map and $u \in [0, 2]$. In the TCT map, u is the parameter and two initial values u_0 and x_0 are needed.

The bifurcation diagram is a straightforward way to show the characteristics of 1D chaotic maps. It plots the distribution of output sequences along with the parameter(s). The chaotic behavior of a 1D chaotic map can be easily seen from its bifurcation diagram.

The bifurcation diagram of the TCT map is shown in Fig. 4. From the diagram, we can see that the TCT map has chaotic behaviors in the whole parameter range, and its output values distribute in a wider ranges compared with its seed map, the Tent map shown in Fig. 1.

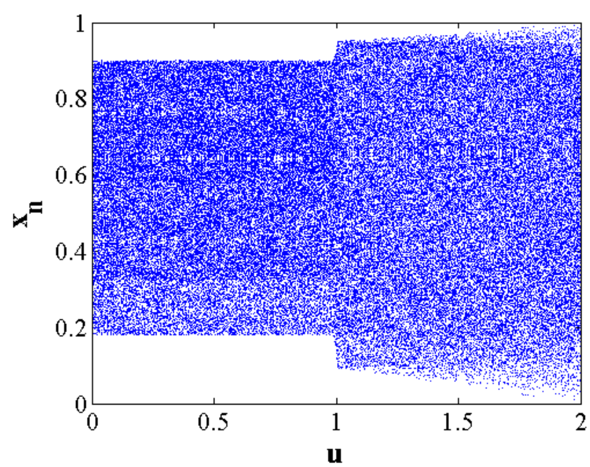

Figure 4: The bifurcation diagram of the TCT map.

4.2 The Logistic-control Sine (LCS) map

When the seed and control maps are two different 1D chaotic maps, and the Logistic map acts as the control map and the Sine map is used as the seed map, a new 1D chaotic map can be generated, called the Logistic-control Sine (LCS) map. The structure of the LCS map is shown in Fig. 5. From the structure we can see that the Logistic map is used to generate dynamic parameters, and the Sine map uses the scaled dynamic parameters to generate iteration output values.

The mathematic representation of the LCS map is defined as Eqn. (10)

$$x_{n+1} = r'_{n+1} \sin(\pi x_n) \tag{10}$$

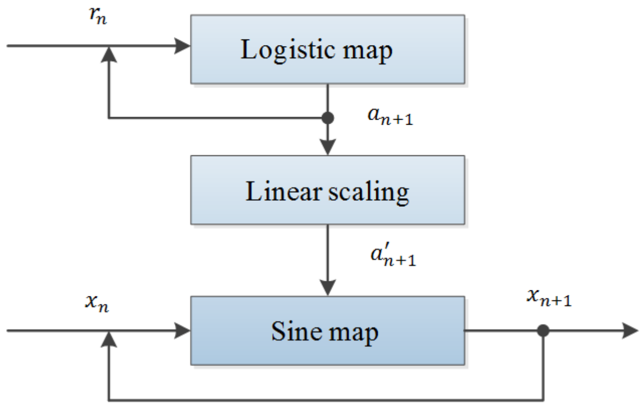

Figure 5: The structure of the LCS map.

in which x_n are iteration input/output values and r'_{n+1} are parameters defined by Eqn. (11)

$$r'_{n+1} = 0.9 + 0.1r_{n+1} \tag{11}$$

where r_{n+1} are the iteration values of the control map, the Logistic map defined by Eqn. (12)

$$r_{n+1} = ar_n(1 - r_n) \tag{12}$$

in which, a is a parameter, r_0 and x_0 are two initial values of the LCS map.

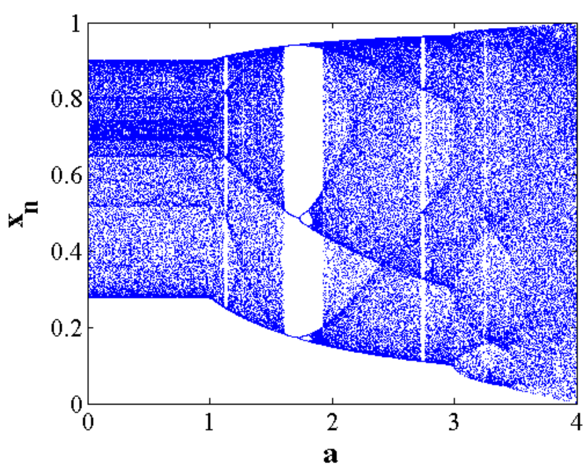

Figure 6: The bifurcation diagram of the LCS map.

The bifurcation diagram of the LCS map is shown in Fig. 6. From the diagram, we can see that the LCS map has chaotic behavior in almost the entire parameter range [0, 4]. For its seed map, the Logistic map, it only has chaotic behaviors in range of [3.57, 4], which can be seen in Fig. 1.

4.3 The Sine-control Logistic (SCL) map

When exchanging the control and seed maps in the LCS map, a new chaotic map can be generated, called the Sine-control Logistic (SCL) map. It uses the Sine map as the control map and the Logistic map as the seed map. The structure of the SCL map can be seen in Fig. 7. From the structure we can see that the Logistic map uses the parameters controlled by the Sine map to generate iteration output values.

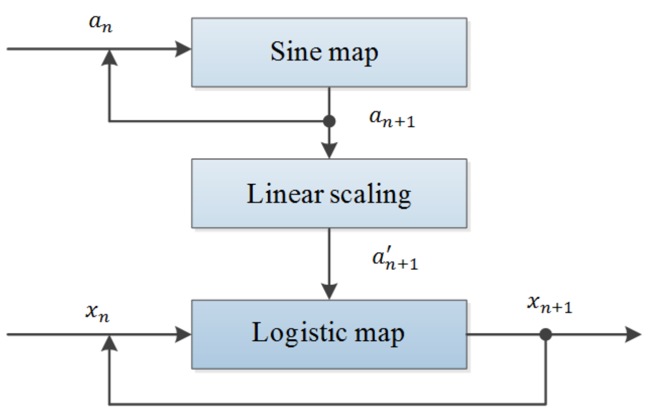

Figure 7: The structure of the SCL map.

The SCL map's mathematic representation is defined by Eqn. (13)

$$x_{n+1} = a'_{n+1}x_n(1 - x_n) \tag{13}$$

where x_n are the iteration input/output values and a'_{n+1} are parameters defined by Eqn. (14)

$$a'_{n+1} = 3.9 + 0.1a_{n+1} \tag{14}$$

where a_{n+1} are the control map's iteration values defined by Eqn. (15)

$$a_{n+1} = r\sin(\pi a_n) \tag{15}$$

where r is a parameter of the control map and $r \in$[0, 1]. In the SCL map, r is the parameter and two initial values a_0 and x_0 are needed.

The bifurcation diagram of the SCL map is shown in Fig. 8. Even the LCS and SCL maps are both generated from the proposed framework using the Logistic and Sine maps, from their bifurcation diagrams in Fig. 6 and Fig. 8, we can see that they are totally different chaotic maps.

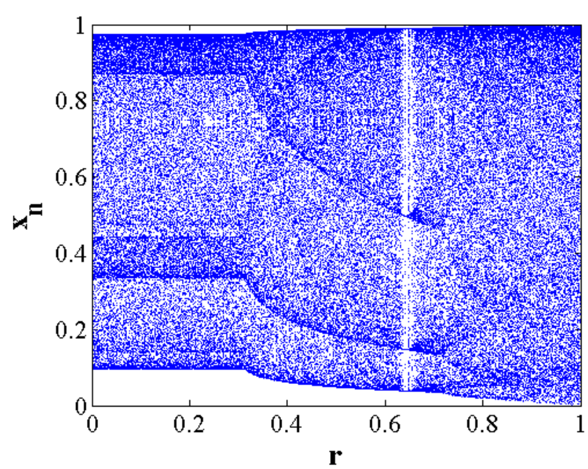

Figure 8: The bifurcation diagram of the SCL map.

5. CHAOS PERFORMANCE ANALYSIS AND COMPARISONS

In this section, we compare chaos performance between the newly generated chaotic maps and their corresponding control and seed maps. The evaluation methods include the Lyaponuv exponent, Information entropy and Correlation test.

5.1 Lyaponuv exponent

The chaotic behaviors of a dynamic system can be quantitatively measured by the Lyapunov exponent.[13] For two extraordinarily close trajectories in the phase plane, the Lyapunov exponent describes the exponential divergences between them. A positive Lyapunov exponent means that the difference between two trajectories will exponentially increase in each unit time. This means that, no matter how small difference their initial values are, the differences between two trajectories will always increase along with the time change, making their output values totally different. That's to say, a dynamic system will be chaotic if it has a positive Lyapunov exponent value and a larger Lyapunov exponent value usually means better chaos performance.

The definition of the Lyapunov exponent for the 1D discrete time system $x_{n+1} = F(x_n)$ is defined by Eqn. (16)

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |F'(x_i)| \tag{16}$$

where $F'(x_i)$ denotes the first-order derivative of $F(x_i)$.

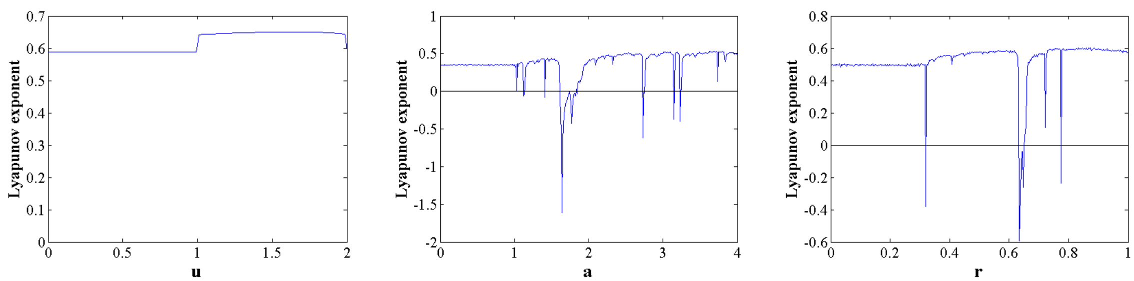

Figure 9: The Lyapunov exponents of new chaotic maps. (a) the TCT map; (b) the LCS map; (c) the SCL map.

The Lyaponuv exponent values of the newly generated 1D chaotic maps are shown in Fig. 9, and the comparison results between these new maps and their corresponding control and seed maps are shown in Table 1. From Fig. 9, we can see that all new chaotic maps have positive Lyapunov exponent values almost in whole ranges of their parameters. This means that they all have wide chaotic ranges. As seen from Table 1, under the same settings of parameters, all new chaotic maps have bigger Lyapunov exponent values than their corresponding control and seed maps. Therefore, new chaotic maps have better chaos performance than their control and seed maps.

Table 1: Comparisons between the new chaotic maps and their seed and control maps.

	Lyaponuv exponent		Information entropy (#Bin:256)	
Parameters (u, r, a)	$(1.3, 0.9, 3.6)$	$(1.5, 0.95, 3.7)$	$(1.3, 0.9, 3.6)$	$(1.5, 0.95, 3.7)$
Tent map (u)	0.1823	0.4055	5.1202	6.5319
Sine map (r)	0.3527	0.4039	6.9830	7.1143
Logistic map (a)	0.1812	0.3544	6.3277	7.0896
TCT map (u)	0.6462	0.6504	7.7901	7.8044
LCS map (a)	0.5075	0.5220	7.8084	7.8464
SCL map (r)	0.5958	0.5871	7.8249	7.8333

5.2 Information entropy

The Information entropy[14] can measure the distribution of a signal or a collection of datas. It gives a quantitative description about the randomness of the signal or datas. The mathematic definition of Information entropy is shown as Eqn. (17)

$$H(X) = -\sum_{i=1}^{n} Pr(x_i)\log_2 Pr(x_i) \tag{17}$$

where X is a collection of data and $Pr(x_i)$ is the probability of the i^{th} possible value in X.

We can use the Information entropy to test the randomness of chaotic sequences. The bigger test value a chaotic sequence gets, the better chaos performance the corresponding chaotic map will have. In our test, we set #Bin:256, which means that we uniformly separate the data of chaotic sequences into 256 different levels. The maximum value of the Information entropy $H(X)_{max} = \text{Log}_2 256 = 8$ if and only if the numbers of data in each level are equal. That means absolutely uniform distribution.

The Information entropy results between new chaotic maps and their control and seed maps are shown in Table 1. Under the same parameter settings, the Information entropy values of the output sequences generated by new chaotic maps are close to the maximum value. They are all bigger than those of the sequences generated by their corresponding control and seed maps. Thus, the new chaotic maps have better chaos performance than their corresponding control and seed maps.

5.3 Correlation test

The relationship between two sequences of data can be described by the correlation. In mathematically, it is defined as Eqn. (18)

$$c = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y} \tag{18}$$

where X and Y are two sequences of data. μ is the mean value and σ is the standard derivation of a sequence.

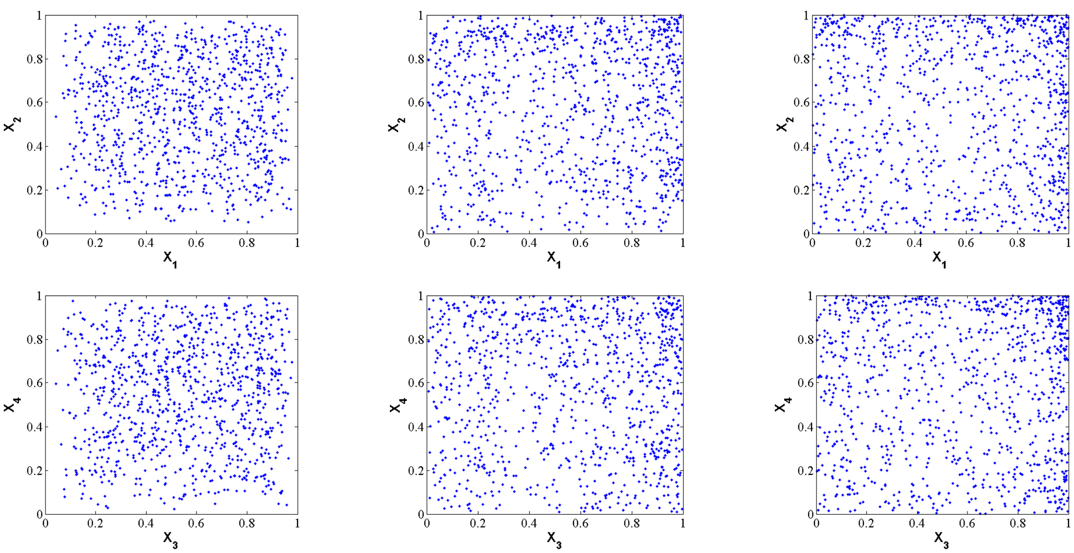

Figure 10: Correlations of sequences generated by different new chaotic maps. The first, second and third columns plot the correlations of the sequences generated by (a) the TCT map; (b) the LCS map and (c) the SCL map with a tiny change in initial values and parameters, respectively.

We can use the correlation to evaluate how a chaotic map is sensitive to its initial values and parameters. When two output sequences are generated by a chaotic map with a tiny change in initial values, smaller absolute correlation value means the chaotic map is more sensitive to its initial values. The same rule happens to the parameter changes.

The relationship between two output sequences generated by new chaotic maps are plotted in Fig. 10, and the correlation test results of new chaotic maps and their control and seed maps are shown in Table 2. Two sequences X_1 and X_2 are generated by chaotic maps with a tiny change (10^{-12}) in initial values, and X_3 and X_4 are generated with a tiny change (10^{-12}) in parameters. As can be seen in Fig. 10, the dots distribute randomly in the whole data ranges. This means that two sequence pairs X_1 and X_2, X_3 and X_4 have no correlation with each other. Then we say that the new chaotic maps are extremely sensitive to their initial values and parameters. From Table 2, we can see that, under the same settings of parameters or initial values. The sequences generated by new chaotic maps have smaller absolute correlation values. Thus, the new chaotic maps are more sensitive to initial values and parameters than their corresponding control and seed maps.

Table 2: Correlation comparisons of the output sequences generated by new chaotic maps and their seed and control maps.

	Correlation of X_1 and X_2		Correlation of X_3 and X_4	
Parameters (u, r, a)	$(1.3, 0.9, 3.6)$	$(1.5, 0.95, 3.7)$	$(1.3, 0.9, 3.6)$	$(1.5, 0.95, 3.7)$
Tent map (u)	0.936523	0.047551	0.935089	0.115970
Sine map (r)	0.127062	0.119240	0.133634	0.012368
Logistic map (a)	0.923856	0.077346	0.926391	0.040122
TCT map (u)	0.051581	-0.015292	0.018896	-0.008228
LCS map (a)	0.022206	0.026636	0.043374	-0.004979
SCL map (r)	0.023941	0.034051	0.007855	-0.009183

6. CONCLUSION

In this paper, a new parameter-control chaotic framework was introduced. In this framework, a 1D chaotic map is used as a seed map while another 1D chaotic map acts as a control map to generate new chaotic maps. Any existing 1D chaotic map can be used as the control or seed map. Three examples of new chaotic maps, including the Tent-control Tent (TCT), Logistic-control Sine (LCS) and Sine-control Logistic (SCL) maps were introduced to show the excellent properties of the proposed framework.

Chaos performance analysis of these newly generated chaotic maps has shown that they can overcome the weaknesses of existing 1D chaotic maps in simple structures and low chaos performance. The comparison results have shown that these new chaotic maps have more complex structures and better chaos performance than their corresponding control and seed maps.

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REFERENCES

[1] Kyrtsou, C. and Labys, W. C., "Evidence for chaotic dependence between us inflation and commodity prices," *Journal of Macroeconomics* **28**(1), 256–266 (2006).

[2] Dilo, R. and Domingos, T., "Periodic and quasi-periodic behavior in resource-dependent age structured population models," *Bulletin of Mathematical Biology* **63**(2), 207–230 (2001).

[3] Zhang, T., Zhou, Y., and Chen, C. L. P., "A new combined chaotic system for image encryption," in [*2012 IEEE International Conference on Computer Science and Automation Engineering (CSAE)*], **2**, 331–335.

[4] Franois, M., Grosges, T., Barchiesi, D., and Erra, R., "A new image encryption scheme based on a chaotic function," *Signal Processing: Image Communication* **27**(3), 249–259 (2012).

[5] Zhou, Y., Panetta, K., Agaian, S., and Chen, C. L. P., "Image encryption using p-fibonacci transform and decomposition," *Optics Communications* **285**(5), 594–608 (2012).

[6] Agaian, S. S., Rudraraju, R. G. R., and Cherukuri, R. C., "Logical transform based encryption for multimedia systems," in [*2010 IEEE International Conference on Systems Man and Cybernetics (SMC)*], 1953–1957.

[7] Wu, Y., Noonan, J. P., and Agaian, S., "Image encryption using the rectangular sudoku cipher," in [*2011 International Conference on System Science and Engineering (ICSSE)*], 704–709.

[8] Zhou, Y. and Agaian, S., "Image encryption using the image steganography concept and plip model," in [*2011 International Conference on System Science and Engineering (ICSSE)*], 699–703.

[9] Sobhy, M. I. and Shehata, A. E. R., "Methods of attacking chaotic encryption and countermeasures," in [*2001 IEEE International Conference on Acoustics, Speech, and Signal Processing*], **2**, 1001–1004, 2001.

[10] Arroyo, D., Rhouma, R., Alvarez, G., Li, S., and Fernandez, V., "On the security of a new image encryption scheme based on chaotic map lattices," *Chaos* **18**(3), 033112–7 (2008).

[11] Wu, Y., Noonan, J. P., and Agaian, S., "A wheel-switch chaotic system for image encryption," in [*2011 International Conference on System Science and Engineering (ICSSE)*], 23–27.

[12] Bao, L., Zhou, Y., Chen, C. L. P., and Liu, H., "A new chaotic system for image encryption," in [*2012 International Conference on System Science and Engineering (ICSSE)*], 69–73.

[13] Li, C.-B., "A chaotic system with constant lyapunov exponent spectrum and its evolvement," in [*2010 International Conference on Measuring Technology and Mechatronics Automation (ICMTMA)*], **1**, 27–31, 2010.

[14] Shannon, C. E., "A mathematical theory of communication," *SIGMOBILE Mobile Computing and Communications Review* **5**(1), 3–55 (2001).